Differential properties of the solution can be studied by utilizing the results obtained. Lemma 4.3 assures an approximate solution of (3.10) according to theorems in [8] on the convergence of the Galerkin method, and of other projection methods.

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Translated by M. D. F.

INTEGRAL CRITERION OF STABILITY FOR SYSTEMS WITH QUASICYCLIC COORDINATES AND ENERGY RELATIONS FOR OSCILLATIONS OF CURRENT-CARRYING CONDUCTORS

PMM Vol. 33, No. 1, 1969, pp. 85-100

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(Received July 25, 1968)

We consider systems with quasicyclic coordinates and analyze the motions in which velocities, impulses and position (but not quasicyclic) coordinates are periodic functions of time. We assume that the generalized forces corresponding to quasicyclic coordinates either depend on time only, or are proportional to quasicyclic generalized coordinates and that the latter are small.

We show that, when certain requirements are imposed on the nonpotential forces with reference to the position coordinates in stable motions, then the quasicyclic impulses assume (up to the small order terms) mean values yielding the minimum of some function Λ of these mean values. This function can be expressed in terms of the Routh's kinetic potential of the system, by the virial describing the forces acting upon the position subsystem by the quasicyclic subsystem, etc. This in turn yields various versions of the integral criterion of stability.

Applying this criterion to the case of the oscillations of linear current-carrying conductors, we can relate mean periodic values of the magnetic fluxes to the extremal conditions of the combination of the averaged values of the magnetic field energy, magnetization energy and of the mechanical kinetic potential (or the virial of the ponderomotive forces).

The case when the Routh's equations are linear with respect to the position coordinates is considered separately, and we refer back to our previous papers on the problems

on excitation of oscillations [1,2] to analyze the possibility of representing the conditions of existence and stability in terms of the harmonic coefficients of action of an oscillating system, and to give specific expressions for Λ.

We thus find that the systems in question constitute a second class of systems admitting the integral criterion. Earlier, Blekhman e.a. [3-5] studied the systems of synchronizable objects with weak constraints. Differences occurring between these two classes are related to the form of the Lagrangian and of the generalized forces as well as to the assumptions on smallness. This leads to different formulations of the criterion. In particular, systems with quasicyclic coordinates, unlike the synchronizable systems, can admit the integral criterion also when considerable dissipation occurs over the position coordinates.

1. Periodic motions in a system with quasicyclic coordinates (*) and the integral criterion of stability. Let a system with holonomic steady constrains be given described by m quasicyclic $(q_1,\ldots,q_m)$ and $n-m$ position $(q_{m+1},\ldots,q_n)$ coordinates, and let the generalized forces corresponding to the quasicyclic coordinates be of two kinds: (1) dependent on time only, or (2) proportional to the quasicyclic generalized velocities. We shall limit ourselves to the case when the forces of the kind (1) are $2\pi/\omega$ periodic and the forces of the kind (2) are small. Consider the motions in which all generalized velocities and impulses as well as the position (but not quasicyclic) coordinates are $2\pi/\omega$ periodic in time. Equations of motion will then be

$$p_r^{\cdot} + \mu h_r q_r^{\cdot} = U_r(t) + \mu f_r \qquad (r = 1,\ldots,m)$$

$$\frac{d}{dt}\frac{\partial L}{\partial q_{m+r}^{\cdot}} - \frac{\partial L}{\partial q_{m+r}} = Q_{m+r} \qquad (r = 1,\ldots,n-m) \tag{1.1}$$

$$L = T(q_{m+1},\ldots,q_n,\ q_1^{\cdot},\ldots,q_n^{\cdot}) - \Pi(q_{m+1},\ldots,q_n)$$

where L is the kinetic potential of the system, p_r $(r = 1,\ldots,m)$ denote quasicyclic impulses, μ is a small parameter and $Q_{m+r}(q_{m+1},\ldots,q_n,\ q_1^{\cdot},\ldots,q_n^{\cdot})$ are nonpotential generalized forces corresponding to the position coordinates.

The most interesting case occurs when "quasicyclic" generalized forces of the second kind are the viscous friction forces and when $\mu h_r > 0$. A more general case when these forces are given in terms of a dissipative function of the form

$$\Phi = \frac{1}{2}\sum_{r,s=1}^{m} h_{rs} q_r^{\cdot} q_s^{\cdot} \tag{1.2}$$

with a positive definite form in its right side can, obviously, be reduced to the previous case by the linear substitution of the quasicyclic coordinates only.

For sufficiently small μ, system (1.1) can have solutions of the type shown above, and they will become solutions of the corresponding generating system when $\mu = 0$ only under the condition that

$$\langle U_r(t)\rangle = 0, \qquad \langle\cdot\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \cdot\, dt \tag{1.3}$$

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*) Following [6], ch. 7, we shall call the coordinates quasicyclic, if they do not appear in the expressions for the kinetic energy and generalized forces, and the corresponding generalized forces are different from zero.

which will, from now on, be assumed; f_r in (1.1) can be assumed constant without loss of generality.

If from the following linear algebraic equations in $q^{\cdot}_r$ $(r = 1,\dots,m)$

$$p_r = \partial T \,/\, \partial q^{\cdot}_r \tag{1.4}$$

we find

$$q_r^{\cdot} = g_r(p_1, \dots, p_m, q_{m+1}, \dots, q_n, \dot{q}_{m+1}, \dots, q_n^{\cdot}) \tag{1.5}$$

and construct the Routh's kinetic potential

$$L_R = \Big[T - \sum_{r=1}^{m} p_r q_r^{\cdot}\Big]_{q_r^{\cdot}=g_r} - \Pi \tag{1.6}$$

then using (1.5) we replace q_r in the expression for Q_{m+r}, the equations of motion can be written in the form

$$\begin{aligned} &p_r^{\cdot} - \mu h_r \frac{\partial L_R}{\partial p_r} = U_r(t) + \mu f_r \qquad (r = 1, \dots, m) \\ &\frac{d}{dt}\frac{\partial L_R}{\partial q_{m+r}} - \frac{\partial L_R}{\partial q_{m+r}} = Q_{m+r} \qquad (r = 1, \dots, n-m) \end{aligned} \tag{1.7}$$

containing only the position coordinates, quasicyclic impulses and their derivatives. For $\mu = 0$ we obtain from (1.7) a system of m equations in quasicyclic impulses

$$p_{r0}^{\cdot} = U_r(t) \qquad (r = 1, \dots, m) \tag{1.8}$$

admitting a family of $2\pi \,/\, \omega$ periodic solutions

$$p_{r0} = \alpha_r + V_r(t) \qquad (r = 1, \dots, m) \tag{1.9}$$

with m arbitrary constants $\alpha_1,\dots\alpha_m$. In (1.9) we have

$$V_r^{\cdot} = U_r, \qquad \langle V_r \rangle = 0 \tag{1.10}$$

We assume that the equations

$$\left[\frac{d}{dt}\frac{\partial L_R}{\partial q^{\cdot}_{m+r}} - \frac{\partial L_R}{\partial q_{m+r}} - Q_{m+r}\right]_{p_r=p_{r0}} = 0 \qquad (r = 1, \dots, n-m) \tag{1.11}$$

for any $\alpha_1,\dots,\alpha_m$ belonging to some region, admit a stable $2\pi/\omega$- periodic isolated solution (i.e. solution without any new constants)

$$q_{r+m} = q_{r+m0}(t, \alpha_1, \dots, \alpha_m) \qquad (r = 1, \dots, n-m) \tag{1.12}$$

Then the equations defining the parameters of the generating solution will be

$$P_r(\alpha_1, \dots, \alpha_m) = -\left\langle \frac{\partial L_R}{\partial p_r} \right\rangle_0 - e_r = 0 \qquad (r = 1, \dots, m) \tag{1.13}$$

The mode corresponding to the solution $\alpha_1 = \alpha_1^*,\dots,\alpha_m = \alpha_m^*$ of (1.13) when μ is sufficiently small will be stable, if the roots $\lambda_1,\dots\lambda_m$ of the equation

$$\det\left\|\left(\frac{\partial P_r}{\partial \alpha_s}\right)_* + \lambda \varkappa_r \delta_{rs}\right\| = 0 \qquad (1.14)$$

have real parts which are negative when $\mu > 0$ and positive when $\mu < 0$. We assume that $\alpha_1^*,\ldots,\alpha_m^*$ belong to the domain of existence of (1.12).

In (1.13), (1.14) and further, the subscript zero will denote the substitution $p_r = p_{r0}$, $q_{r+m} = q_{r+m0}$; the asterisk - the substitution $\alpha_1 = \alpha^*_1,\ldots,\alpha_m = \alpha^*_m$; $e_r = f_r/h_r$, $\varkappa_r = 1/h_r$, and δ_{rs} will denote the Kronecker delta.

Next we shall find the conditions for

$$P_r = -\frac{\partial}{\partial \alpha_r}\left[\langle L_R\rangle_0 + \sum_{r=1}^{m} e_r\alpha_r\right] \qquad (r = 1,\ldots,m) \qquad (1.15)$$

to hold. Integrating by parts, we obtain

$$\left\langle \frac{\partial L_R}{\partial p_r}\right\rangle_0 = \frac{\partial}{\partial \alpha_r}\langle L_R\rangle_0 + \sum_{s=1}^{n-m}\left\langle \left[\frac{d}{dt}\frac{\partial L_R}{\partial \dot q_{m+s}} - \frac{\partial L_R}{\partial q_{m+s}}\right]_0 \frac{\partial q_{m+s0}}{\partial \alpha_r}\right\rangle \qquad (1.16)$$

From (1.7) it follows that (1.15) hold when

$$\sum_{s=1}^{n-m}\left\langle Q_{m+s0}\frac{\partial q_{m+s0}}{\partial \alpha_r}\right\rangle = 0 \qquad (r = 1,\ldots,m) \qquad (1.17)$$

Equations (1.17) imply that the matrix $\|\partial P_r / \partial \alpha_s\|$ is symmetric. Let us limit ourselves to the case when the quasicyclic generalized forces of the second kind are the viscous friction forces, assuming also that $\mu > 0$ and $h_r > 0$ $(r = 1,\ldots,m)$. We shall, in addition, use the following fact.

Let the $N \times N$ matrices A and B be symmetric and let B be positive definite. Let also $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_N$ and $\lambda_1' \leqslant \lambda_2' \leqslant \ldots \leqslant \lambda_N'$ be the roots of the equations $|A - \lambda E| = 0$ and $|A - \lambda' B| = 0$ respectively, where E is a unit $N \times N$ matrix. Then λ_i and λ_i' have the same sign.

Since all $\varkappa_r > 0$, the matrix $\mathrm{diag}\,(\varkappa_1,\ldots,\varkappa_m)$ is positive definite and, when $\|\partial P_r / \partial \alpha_s\|$ is symmetric, the stability will depend on the signs of the roots $\lambda_1',\ldots,\lambda_m'$ of the equation

$$\det\|(\partial P_r / \partial \alpha_s)_* + \lambda'\delta_{rs}\| = 0 \qquad (1.18)$$

Moreover, the point $(\alpha_1^*,\ldots,\alpha_m^*)$ will be a stationary point of the function

$$\Lambda(\alpha_1,\ldots,\alpha_m) = -\langle L_R\rangle_0 - \sum_{r=1}^{m} e_r\alpha_r \qquad (1.19)$$

therefore the mode corresponding to the values $\alpha_1 = \alpha_1^*,\ldots,\alpha_m = \alpha_m^*$ will be stable, if the above-mentioned point is a minimum.

The latter statement forms a basis for the integral criterion of stability for the class of systems under consideration. The criterion will certainly hold, if all generalized forces written in terms of the position coordinates are potential forces. It is nevertheless true, that the integral criterion may hold also when nonpotential generalized forces

Q_{m+s} are present.

Let e.g. Q_{m+s} be expressed as linear forms of the position velocities

$$Q_{m+s} = \sum_{r=1}^{n-m} b_{rs} q^{\cdot}_{m+r} \qquad (s = 1, \ldots, n-m) \tag{1.20}$$

and q_{m+s0} be represented by series of the form

$$q_{m+s0}(t, \alpha_1, \ldots, \alpha_m) = \sum_{\nu} q^{(\nu)}_{m+s0}(\alpha_1, \ldots, \alpha_m) \cos(\nu\omega t - \varphi_\nu) \tag{1.21}$$

where the phase shifts φ_ν of the harmonic components of q_{m+s0} are independent of $\alpha_1, \ldots, \alpha_m$ and are identical for all $q_{m+10}, \ldots, q_{n0}$. Then

$$\sum_{s=1}^{n-m} \left\langle Q_{m+s0} \frac{\partial q_{m+s0}}{\partial \alpha_r} \right\rangle = \sum_{s=1}^{n-m} \sum_{l=1}^{n-m} b_{ls} \left\langle - \sum_{\nu} \nu\omega q^{(\nu)}_{m+l0} \times \right.$$

$$\left. \times \sin(\nu\omega t - \varphi_\nu) \sum_{\nu} \frac{\partial q^{(\nu)}_{m+s0}}{\partial \alpha_r} \cos(\nu\omega t - \varphi_\nu) \right\rangle = 0 \qquad (r = 1, \ldots, n-m) \tag{1.22}$$

When all components in the Fourier expansion of a function exhibit the same phase shift, we shall say that this function is component-wise phase-coupled and we shall call the conditions that $\varphi_{m+10}^{(\nu)} = \ldots = \varphi_{n0}^{(\nu)} = \varphi_\nu$ - the conditions of the component-wise phase-coupling. We can now say that the sufficient condition for the integral criterion to exist when Q_{m+s} are linear forms of $q^{\cdot}_{m+1}, \ldots, q_n^{\cdot}$ is, that the phase shifts in the expansions of the position coordinates computed for the generating approximation are independent of the parameters of the generating solution and that these coordinates satisfy the condition of the component-wise phase coupling. Since no constraints of any sort are imposed on the properties of the matrix $\|b_{rs}\|$, it follows that the integral criterion exists in the case when Q_{m+s} represent the viscous friction forces.

The case which we have just discussed has no analog in the problems on synchronization, since in the latter case $\alpha_1, \ldots, \alpha_m$ represent the phase shifts of the object coordinates [3-5], expressions $\omega t + \alpha_r$, and $\varphi_{m+r0}^{(\nu)}$ appearing in the solutions most certainly depend on $\alpha_1, \ldots, \alpha_m$ and the integral criterion is possible only in the absence of dissipation in the supporting (oscillating) system [3-5].

The equation

$$L_R = T_2 - \Pi - T_1 = L_2 - T_1 \tag{1.23}$$

where T_1 and T_2 are given by

$$T = T_1 + T_2 + U \tag{1.24}$$

$$T_1 = \frac{1}{2} \sum_{r,s=1}^{m} A_{rs} q_r^{\cdot} q_s^{\cdot},$$

$$T_2 = \frac{1}{2} \sum_{r,s=1}^{n-m} A_{m+r\, m+s} \dot{q}_{m+r} \dot{q}_{m+s}, \qquad U = \sum_{r=1}^{m} \sum_{s=1}^{n-m} A_{r\, m+s} q_r^{\cdot} q^{\cdot}_{m+s}$$

enables us to obtain Λ in a more definite form

$$\Lambda = \langle T_1 \rangle_0 - \langle L_2 \rangle_0 - 2A, \qquad A = \frac{1}{2} \sum_{r=1}^{m} e_r \alpha_r \tag{1.25}$$

which will be preserved in the case when the left parts in (1.17) are equal to b_r, and are independent of $\alpha_1, \ldots, \alpha_m$. At the same time e_r in (1.25) should be replaced with $e_r - b_r$ (see Sect. 3).

Two remarks which follow refer to the practical applications in connection with the problems on excitation of mechanical oscillations.

The generating approximation and the formulation of the integral criterion are not affected by the addition of any terms of the form μ ( ) to the first m equations of (1.1) and of any terms of the order μ to the remaining $n - m$ equations. The form of $\langle L_R \rangle_0$ as a function of $\alpha_1, \ldots, \alpha_m$ remains the same even when new degrees of freedom appear corresponding to the group of coordinates $q_{n+1}, \ldots, q_{n+l}$ such that the expression for the kinetic energy becomes

$$T = T_1 + T_2 + U + T_*, \qquad T_* = T_*(q_{n+1}, \ldots, q_{n+l}, \dot{q}_{n+1}, \ldots, \dot{q}_{n+l})$$

$$T_1 = \frac{1}{2} \sum_{r,s=1}^{m} A_{rs} \eta_r^{\cdot} \eta_s^{\cdot}, \qquad U = \sum_{r=1}^{m} \sum_{s=1}^{n-m} A_{r\,m+s} \eta_r^{\cdot} \dot{q}_{m+s} \tag{1.26}$$

Coefficients of the forms T_1 and U depend only on $q_{m+1}, \ldots, q_n$ and T_2 is given in (1.24)

$$\eta_r = q_r + \sum_{i=1}^{l} w_{ri} q_{n+i} \qquad (w_{ri} = \text{const}) \tag{1.27}$$

We assume that the generalized forces in $q_{n+1}, \ldots, q_{n+l}$ are independent of the remaining generalized coordinates and velocities and that the forces are $2\pi / \omega$ periodic in time, which appears in the formulas explicitly.

To clarify the form of the corresponding Routh's equations, we shall write the kinetic energy as

$$T = T_1^{(1)} + U_{1*} + T_{1*} + U^{(1)} + U_* + T_* + T_2 \tag{1.28}$$

where

$$T_1^{(1)} = \frac{1}{2} \sum_{r,s=1}^{m} A_{rs} q_r^{\cdot} q_s^{\cdot}, \qquad U^{(1)} = \sum_{r=1}^{m} \sum_{s=1}^{n-m} A_{r,\,m+s} q_r^{\cdot} \dot{q}_{m+s}$$

$$U_{1*} = \sum_{r,s=1}^{m} A_{rs} q_r^{\cdot} \dot{q}_{s*}, \qquad U_* = \sum_{r=1}^{m} \sum_{s=1}^{n-m} A_{r,\,m+s} \dot{q}_{r*} \dot{q}_{m+s}$$

$$T_{1*} = \frac{1}{2} \sum_{r,s=1}^{m} A_{rs} \dot{q}_{r*} \dot{q}_{s*}, \quad q_{r*} = \eta_r - q_r \qquad (r = 1, \ldots, m) \tag{1.29}$$

We have

$$q_r^{\cdot} = g_r - \dot{q}_{r*} \qquad (r = 1, \ldots, m) \tag{1.30}$$

where g_r are functions of $p_1, \ldots, p_m, q_{m+1}, \ldots, q_n, \dot{q}_{m+1}, \ldots, \dot{q}_n$, are given by (1.5). The form of g_r remains the same as that for the system without additional coordinates.

Routh's kinetic potential will be given by

$$L_R = -T_1^{(1)} + T_{1*} + U_* + T_* + T_2 - \Pi \tag{1.31}$$

Performing the substitution according to (1.30) for $\dot{q}_r$ $(r = 1,\ldots,m)$ in $T_1^{(1)}$ and using the identity

$$\sum_{r,s=1}^{m} A_{rs} g_s q_{r*} + \sum_{r=1}^{m} \sum_{s=1}^{n-m} A_{rm+s} \dot{q}_{r*} \dot{q}_{m+s} = \sum_{r=1}^{m} p_r \dot{q}_{r*} \tag{1.32}$$

we obtain

$$L_R = L_R^{(0)} + \sum_{r=1}^{m} p_r \dot{q}_{r*} + T_* \tag{1.33}$$

where $L^{(0)}_R$ is the Routh's kinetic potential for the system without additional coordinates. We therefore see that generating equations for $p_1,\ldots,p_m$ and $q_{m+1},\ldots,q_n$ obtained from (1.7) have the same form and the same solutions as those for the system without additional coordinates. Equations corresponding to the additional coordinates will be

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot{q}_{n+i}} - \frac{\partial T_*}{\partial q_{n+i}} = -\sum_{r=1}^{m} w_{ri} \dot{p}_r + Q_{n+i} \quad (i = 1,2,\ldots,l) \tag{1.34}$$

(these equations could have been written down at once as the Lagrange s equations). System (1.34) contains no $\alpha_1,\ldots,\alpha_m$ in its generating approximation. We assume that when the substitution $\dot{p}_r = U_r$ is performed, then the system will admit an isolated stable solution in which $\dot{q}_{n+10},\ldots,\dot{q}_{n+l0}$ are $2\pi/\omega$ periodic functions of time. Then the equations defining $\alpha_1,\ldots,\alpha_m$ and the conditions of stability will differ from those occurring in the system without additional coordinates only in the values of e_r which will become

$$e_r^* = e_r + \sum_{i=1}^{l} w_{ri} \langle \dot{q}_{n+i0} \rangle \tag{1.35}$$

Although T_1 is a sign definite form of quasicyclic and supplementary velocities only, the above remark has a definite physical sense (see Sect. 2).

The above results can be extended to rotational motions when $q_{m+s} = \omega t + 2\pi/\omega$-periodic function for certain position coordinates. At the same time, functions T, Π and Q_{m+s} should be $2\pi/\omega$- periodic in the corresponding q_{m+s} or contain only their differences $q_{m+r} - q_{m+s}$.

Let us also consider the case when L_2 corresponds to a linear system. Then Routh's equations over the position coordinates will be

$$M\ddot{u} + Cu = Q + F, \qquad F = -\frac{\partial T_1}{\partial u} \tag{1.36}$$

where $u = (q_{m+1},\ldots,q_n)$, $Q = (Q_{m+1},\ldots,Q_n)$; M and C are symmetric $(n-m) \times (n-m)$ matrices with constant components. Denoting the scalar products by brackets, we obtain

$$L_2 = {}^1/_2\,(M\dot{u}, \dot{u}) - {}^1/_2\,(Cu, u) \tag{1.37}$$

Let us scalar multiply both parts of the equation (1.36) written in its generating approximation by $\partial u_0 / \partial \alpha_r$ and average the result over one period. Assuming that (1.17) holds, we obtain

$$\left\langle\left(Mu_0, \frac{\partial u_0}{\partial \alpha_r}\right)\right\rangle + \left\langle\left(Cu_0, \frac{\partial u_0}{\partial x_r}\right)\right\rangle = \left\langle\left(F_0, \frac{\partial u_0}{\partial x_r}\right)\right\rangle \qquad (1.38)$$

On the other hand, differentiating both parts of the same equation with respect to α_r, scalar multiplying the result by u_0 and averaging, we find

$$\left\langle\left(M\frac{\partial u_0}{\partial \alpha_r}, u_0\right)\right\rangle + \left\langle\left(C\frac{\partial u_0}{\partial x_r}, u_0\right)\right\rangle = \left\langle\left(\frac{\partial Q_0}{\partial x_r}, u_0\right)\right\rangle + \left\langle\left(\frac{\partial F_0}{\partial x_r}, u_0\right)\right\rangle \qquad (1.39)$$

Since the matrices M and C are symmetric, integration by parts of the first terms of (1.38) and (1.39) yields

$$\left\langle\left(F_0, \frac{\partial u_0}{\partial \alpha_r}\right)\right\rangle = \frac{\partial}{\partial \alpha_r}\langle(Q_0, u_0)\rangle + \left\langle\left(\frac{\partial F_0}{\partial x_r}, u_0\right)\right\rangle - $$
$$- \frac{\partial}{\partial \alpha_r}\langle L_2\rangle_0 = \left\langle\left(F_0, \frac{\partial u_0}{\partial \alpha_r}\right)\right\rangle \qquad (1.40)$$

from which we obtain

$$\frac{\partial}{\partial x_r}\langle L_2\rangle_0 = \frac{1}{2}\frac{\partial}{\partial x_r}(W_{Q_0} + W_{F_0}^*) \qquad (1.41)$$

where W_Q and W_F are the virials of the nonpotential forces corresponding to other position coordinates and of the forces of action of the "quasicyclic subsystem" on the "position" (oscillating) subsystem. These virials are defined by

$$W_Q = -\langle(Q, u)\rangle, \qquad W_F = -\langle(F, u)\rangle \qquad (1.42)$$

Relations (1.41) enable us to eliminate, in the given case, $\langle L_2\rangle_0$ from the expression for Λ

$$\Lambda = \langle T_1\rangle_0 - {}^1/_2 W_{Q_0} - {}^1/_2 W_{F_0} - 2A \qquad (1.43)$$

If $Q_{m+1},\ldots,Q_n$ are linear forms of $\dot q_{m+1},\ldots,\dot q_n$ and $q_{m+10},\ldots,q_{n0}$ are componentwise phase-coupled and phase independent of $\alpha_1,\ldots,\alpha_m$, then $W_{Q0} = 0$, (1.17) holds and

$$\Lambda = \langle T_1\rangle_0 - {}^1/_2 W_{F_0} - 2A \qquad (1.44)$$

Equations $W_{Q_0} = 0$ and (1.17) hold in certain other cases (see Sect. 3). Representations of Λ in the form (1.44) which are possible in the problems on the excitation of oscillations, yield the following. Let u and $q_1,\ldots,q_m$ be coordinates of the oscillating system and of the exciters [1, 2]. In (1.24) we have

$$T_1 = T_1(\xi, \dot q_1,\ldots,\dot q_m), \qquad U = U(\xi, \dot\xi, \dot q_1,\ldots,\dot q_m)$$
$$\xi = (\xi_1,\ldots,\xi_k), \qquad \xi_j = (u, v_j) \qquad (j = 1,\ldots,k) \qquad (1.45)$$

where $\xi_1,\ldots,\xi_k$ are the feedback parameters [1,2] and v_j are constant $(n-m)$-vectors. The form of T_1 and U does not depend on the form of the oscillatory system [1] (i.e. on the number of its degrees of freedom, on the manner of introduction of the coordinates u etc.; these factors only define the form of v_j). This gives

$$F=\sum_{j=1}^{k}\left(\frac{d}{dt}\frac{\partial T_1}{\partial \xi_j^{\cdot}}-\frac{\partial T_1}{\partial \xi_j}\right)v_j,\qquad T_1=T_1(\xi,\ \xi^{\cdot},\ p_1,\ldots,p_m)$$

$$W_F=\left\langle-\sum_{j=1}^{k}\left(\frac{d}{dt}\frac{\partial T_1}{\partial \xi_j^{\cdot}}-\frac{\partial T_1}{\partial \xi_j}\right)\xi_j\right\rangle \tag{1.46}$$

Consequently, when Λ is defined by (1.44), the form of the averaged functions of ξ, $\xi^{\cdot}$, and $p_1,\ldots,p_m$ will not depend on the form of the oscillating system.

2. Energy relations for the case of oscillation of the current-carrying conductors. Let a solid body system be given, incorporating m linear conductors to which known external $2\pi/\omega$-periodic emf's are applied. We shall consider the case when the electric field energy outside the conductors can be neglected and the magnetic field can be assumed quasi-stationary [7] within the range of frequencies given by $\omega,\ldots,\nu_*\omega$, where ν_* is sufficiently large. (In general, the arguments which follow are valid only to within the high frequency "tails" of the functions to be determined, beginning with some harmonic frequency ν_*+1. This is connected with the fact that the dynamic effects in the material are neglected e.a.). We assume that the relationship between $\mathbf{B}$ and $\mathbf{H}$ in the material is linear and that the resistances of the conductors are small compared with the inductive reactances at the frequency ω.

Since $\mathbf{B}$ and $\mathbf{H}$ are connected linearly, we can describe the system in terms of Lagrangian equations, supplementing the kinetic energy with the part of the total free energy depending on the currents (magnetic field energy) W

$$W=\frac{1}{2}\sum_{r,s=1}^{m}L_{rs}^{*}i_r i_s \tag{2.1}$$

Here $L_{rr}^{*}=L_{rr}^{*}(q_{m+1},\ldots,q_n)$ and $L_{rs}^{*}=L_{rs}^{*}(q_{m+1},\ldots,q_n)$ are coefficients of the self- and the mutual induction, $i_r=q_r$ $(r=1,\ldots,m)$ denote the currents in the conductors and $q_{m+1},\ldots,q_n$ are the mechanical generalized coordinates. Coordinates (charges) q_r are quasicyclic.

Equations of motion will be

$$\begin{gathered}\Phi_r+\mu R_r i_r=U_r(t)+\mu f_r \qquad (r=1,\ldots,m)\\ \frac{d}{dt}\frac{\partial L_2}{\partial \dot q_{m+r}}-\frac{\partial L_2}{\partial q_{m+r}}=Q_{m+r}+F_{m+r}\qquad (r=1,\ldots,n-m)\end{gathered} \tag{2.2}$$

where

$$F_{m+r}=\frac{\partial}{\partial q_{m+r}}W(i_1,\ldots,i_m,q_{m+1},\ldots,q_n) \tag{2.3}$$

are the ponderomotive forces and

$$\Phi_r = \frac{\partial}{\partial i_r} W(i_1, \ldots, i_m, q_{m+1}, \ldots, q_n) \tag{2.4}$$

denote the fluxes of magnetic induction across the conductors.

In (2.2), μR_r denote the resistances of the conductors; U_r and μf_r the alternating and direct components of the applied emf, respectively, (the latter should be small so that no currents $i = O(1/\mu)$ are present in the stationary mode).

Parameters $\alpha_1, \ldots, \alpha_m$ of the generating solution here play the part of the direct components of magnetic fluxes computed with the accuracy of up to the small terms

$$\Phi_{r0} = \alpha_r + V_r(t) \qquad (r = 1, \ldots, m) \tag{2.5}$$

The quantity

$$2A = \sum_{r=1}^{m} e_r \alpha_r, \qquad e_r = \frac{f_r}{R_r} \tag{2.6}$$

in this case, is the energy of magnetization and has the following sense. Let all $U_r = 0$ and let a constant flux α_r pass through the r-th circuit. Then the current in the r-th circuit will be given by $i_r = e_r$ and the energy of this system of constant currents (magnetization currents) in the given field, will be equal to $2A$ (the field is assumed external to the currents, see [7], ch. IV, Sect. 32, 32.14).

Relation (1.25) enables us to formulate the following statement. If nonpotential mechanical forces are absent from the system or, if these forces satisfy (1.17), then when a stable periodic motion (*) occurs, the direct components of magnetic fluxes (with the accuracy of up to the small terms) will have such values that the function of these components will be a minimum, its value equal to the mean (over one period) value of the magnetic field energy, less the mean (over one period) value of the mechanical kinetic potential and the energy of magnetization.

In the case when L_2 corresponds to a linear oscillating system, the mechanical kinetic potential in the integral criterion can be replaced (in accordance with (1.43)) with the half-sum of the virials of the nonpotential mechanical and ponderomotive forces.

We shall, in addition, assume that other linear conductors are situated near the ones under consideration in such a manner than when a line of magnetic induction envelops a "primary" conductor, then it must envelop the whole group of secondary conductors situated near the "primary". We assume that the resistances of the secondary conductors are not small (otherwise the currents would be $i = O(1/\mu)$). Then the charges carried across the secondary conductors will play the part of the additional coordinates in accordance with Sect. 1, and the quantities η_r will be given by

$$\eta_r = q_r + \sum_{j=1}^{l_r} w_{rj} q_j^{(r)} \tag{2.7}$$

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*) More accurately - a motion existing at sufficiently small μ and described by a solution which becomes the corresponding solution of the generating system when $\mu = 0$; moreover, the currents and displacements are periodic in these motions, the charges are not.

where l_r is the number of the secondary conductors situated near the r-th primary, $l_1 + \ldots + l_m = l$, $q_j^{(r)}$ is the charge which has passed the plane of cross section of the j-th conductor of the r-th group and w_{rj} are rational numbers defined as follows. Let the circuit of the r-th primary conductor pass along a certain line $n_1^{(r)}$ times, and the circuit of the $j^{(r)}$-th conductor pass $n_j^{(r)}$ times. Then $w_{rj} = n_j^{(r)} / n_1^{(r)}$ (in practice $n_j^{(r)}$ denotes the number of turns).

The effect exerted by these secondary conductors on the motion of the system does not depend on the manner of connections made between these conductors, nor on the emf applied, nor on the character of other electrical components included in the network (such as coils, condensers, rectifiers, e.a.). It is indicated in terms of e_r^* only, the latter replacing e_r in (2.6)

$$e_r^* = e_r + \sum_{j=1}^{l_r} w_{rj} \langle i_{j0}^{(r)} \rangle, \qquad i_j^{(r)} - (q_j^{(r)}). \tag{2.8}$$

Since lines of induction enveloping the primary conductor and not the secondary ones always exist, the above argument is valid only under the condition that the "difference" can be described using terms of the order of μ in the expression for W. Then, additional terms of the form $\mu(\ )^{\cdot}$ will appear in the first m equations of (2.2) for the mechanical coordinates, and they will not alter the generating solution.

3. Routh's equations linear in position coordinates. Even if L_2 corresponds to a linear system, equations defining the position coordinates in the generating approximation will, generally speaking, be still nonlinear, since the forces F depend on the position coordinates. Two cases, however, exist in which linear equations are obtained. When the constraints are stationary, the Routh's function $R = L_R + \Pi$ has the following structure

$$R = \frac{1}{2} \sum_{r,s=1}^{n-m} (A_{m+rm+s} + N^{*}_{m+rm+s}) \dot{q}_{m+r} \dot{q}_{m+s} + \\ + \sum_{r=1}^{n-m} \sum_{s=1}^{m} N_{m+rs} p_s \dot{q}_{m+r} - \frac{1}{2} \sum_{r,s=1}^{m} A^{(rs)} p_r p_s \tag{3.1}$$
$$(\|A^{rs}\| = \|A_{rs}\|^{-1})$$

and the forces of action of the quasicyclic system on the position system will be

$$F_{m+r} = -\frac{d}{dt} \sum_{s=1}^{n-m} N_{m+rm+s} \dot{q}_{m+s} + \frac{1}{2} \sum_{i,s=1}^{n-m} \frac{\partial_{m+im+s}}{\partial q_{m+r}} \dot{q}_{m+i} \dot{q}_{m+s} - \\ - \sum_{s=1}^{m} p_s \sum_{i=1}^{n-m} \dot{q}_{m+i} \left(\frac{\partial N_{m+rs}}{\partial q_{m+i}} - \frac{\partial N_{m+is}}{\partial q_{m+r}} \right) - \sum_{s=1}^{m} N_{m+rs} \dot{p}_s - \frac{1}{2} \sum_{i,s=1}^{m} \frac{\partial A^{(is)}}{\partial q_{m+r}} p_i p_s$$
$$(r = 1, \ldots, n-m) \tag{3.2}$$

Since our aim is to obtain linear equations for the position coordinates in the generating approximation, we shall assume that nonpotential forces written in terms of the position coordinates are linear forms of $\dot{q}_{m+1}, \ldots, \dot{q}_n$ with constant coefficients. From (3.2) it follows that two cases are possible.

1. Quasiharmonic generating system. Quasiharmonic equations (i.e. linear equations with periodic coefficients) are obtained if $N_{m+rm+s} = \text{const}$ $(r, s = 1,\ldots, n-m)$, N_{m+rs} $(r = 1,\ldots, n-m;\ s = 1,\ldots,m)$ are sums of constant quantities and linear forms; $A^{(rs)}$ $(r, s = 1,\ldots,m)$ are sums of constant quantities and linear and quadratic forms of the position coordinates. Constant terms in $A^{(rs)}$ do not affect the form of F_r, and the second term in the right side of (3.2) vanishes. System of the generating equations will be inhomogeneous if at least one of the following conditions holds: (a) N_{m+rs} have constant terms or (b) $A^{(rs)}$ have linear terms. If all N_{m+rs} are linear forms and $A^{(rs)}$ have the form: constant plus quadratic form, the system will be homogeneous.

At this point we shall turn our attention to one particular case. Let the products of the quasicyclic and position velocities be absent from the expression for kinetic energy ($U = 0$). Then all $N_{m+rs} = 0$, and $N_{m+rm+s} = 0$. Let, in addition, $A^{(rs)}$ contain no linear terms. Decomposing T_1 into the energy of the quasicyclic subsystem with T_1^* restrained, and the "additional energy" ΔT_1

$$T_1 = T_1^* + \Delta T_1, \quad T_1^* = \frac{1}{2}\sum_{r,s=1}^{m} A_*^{(rs)} p_r p_s, \quad \Delta T_1 = \frac{1}{2}\sum_{r,s=1}^{m} \Delta A^{(rs)} p_r p_s \tag{3.3}$$

where $\Delta A^{(rs)}$ are quadratic forms in q_{m+r}, we obtain

$${}^1/_2 W_F = \Delta T_1 \tag{3.4}$$

Function Λ will, at the same time, have the form

$$\Lambda = \langle T_1^* \rangle_0 - {}^1/_2 W_{Q_0} - 2.4 \tag{3.5}$$

If, in addition, $W_{Q0} = 0$, then Λ will assume the form which it has when $q_{m+1}, \ldots, q_n \equiv 0$, i.e. when the position subsystem is restrained. Consequently, in this case the position subsystem does not (within the accuracy of up to the small terms) influence the motion of the quasicyclic subsystem; the motions of the position subsystem, however, are substantially dependent on the quasicyclic subsystem. Applying this to the problems on excitation of oscillations we see that it means that the feedback action of the oscillation on the exciter is insignificant despite the presence of a family of the generating solutions and the fact that small terms depending on the position coordinates are substantial.

If $T_1^* = 0$ and at least one $f_r \neq 0$, then no solutions of the type discussed above exist. If, on the other hand, all $f_r = 0$, then we arrive at the singular case of the method of small parameters ($P_r \equiv 0$) in which terms of the order of μ^2 must be taken into account in the required solutions.

2. Generating system with constant coefficients. Equations with constant coefficients are obtained if $N_{m+rs} = \text{const}, N_{m+rm+s} = \text{const}$, and $A^{(rs)}$ are linear forms of the position coordinates. In this case position coordinates in the generating approximation are defined from the solution of the problems on forced oscillations of a linear system acted upon by the forces which can be expressed in terms of some known functions of time and of the parameters $\alpha_1,\ldots,\alpha_m$.

This system, however, will differ from the initial oscillating system (with the kinetic potential L_2) because of the terms $N_{m+rm+s} q_{m+r}$. Action of the quasicyclic subsystem on the position subsystem, within the accuracy of up to small terms, thus consists of the following: firstly, it is responsible for $2\pi/\omega$-periodic driving force, and secondly, it modifies the masses of the oscillating system. Rigid constraints however are not altered and the gyroscopic forces do not appear.

The most interesting case in the problems on the excitation of mechanical oscillations is that which occurs when $U = 0$ We then have in the generating approximation, the problem on the forced oscillations of the initial oscillating system. If at the same time we can write the expression for T_1 so that it contains the feedback parameters (i.e. in the form invariant with respect to the type of the oscillating system [1,2]), then equations defining the parameters of the generating solution and the conditions of stability, can be written so that they contain harmonic coefficients of action of the oscillating system as parameters. If this holds for some exciter, then the oscillations caused by this exciter in any linear mechanical system will be fully defined if the coefficients of action for this system are found and the relations shown above utilized. Below we shall determine the form of p_r and Λ for such cases.

Expression for the Routh's kinetic potential will be (*)

$$L_R = L_2 - \frac{1}{2} \sum_{r,s=1}^{m} \left[A_*^{(rs)} + \sum_{j=1}^{k} \Delta A_j^{(rs)} \xi_j \right] p_r p_s = L_2 - T_1^* - \Delta T_1 \quad (3.6)$$

and the corresponding expression for the driving forces,

$$F = \sum_{j=1}^{k} F_j v_j, \qquad F_j = -\frac{1}{2} \sum_{r,s=1}^{m} \Delta A_j^{(rs)} p_r p_s \quad (3.7)$$

Notations in (3.6) and (3.7) correspond to those of (1.45) and (1.46); the number k of the feedback parameters and the coefficients of $A^{(rs)}$ and $\Delta A_j^{(rs)}$ follow from the properties of the exciter only.

Driving forces computed in the generating approximation will be functions of the parameters $\alpha_1, \ldots, \alpha_m$ of the generating solution and of time only. They are obtained by inserting (1.9) into (3.7)

$$F_{j0} = -\frac{1}{2} \sum_{r,s=1}^{m} \Delta A_j^{(rs)} (\alpha_r \alpha_s + 2\alpha_r V_s + V_r V_s) \quad (3.8)$$

Let us now bring into consideration, in accordance with [1,2] , harmonic coefficients of the action of the oscillating system $k_\nu^{(ij)}$ and the phase shifts $\psi_\nu^{(ij)}$ which are determined in the following manner.

Let the oscillating system be acted upon by a single given load of the form $v_i \cos \nu\omega t$. We shall define the resulting pure forced ($2\pi / \nu\omega$-periodic) oscillations and find the laws governing the behavior of the feedback parameters over a period of time. We denote their amplitudes and phase shifts relative to the load by $k_\nu^{(ij)}$. and $\psi_\nu^{(ij)}$, respectively, in accordance with

$$\xi_j = k_\nu^{(ij)} \cos (\nu\omega t - \psi_\nu^{(ij)}) \qquad (j = 1, \ldots, k) \quad (3.9)$$

*) In a number of cases the form of L_R differs from that assumed by the small terms responsible for insignificant additions of the order of μ to the equations of motion of the oscillating system. Moreover, the term under the summation sign in (3.6) should represent a positive definite form of quasicyclic impulses. This leads to the necessity of imposing constraints on ξ_j , of the form $\Delta_j < \xi_j < \Delta_{j*}$ which, in most cases, are obvious from the physical sense of a given problem.

This enables us to obtain in the generating approximation the feedback parameters as functions of time and of $\alpha_1, \ldots, \alpha_m$, as well as of $k_\nu^{(ij)}$ and $\psi_\nu^{(ij)}$

$$\xi_{j0} = -\frac{1}{2}\sum_{i=1}^{k}\Big[\sum_{r,s=1}^{m}\Delta A_i^{(rs)}k_0^{(ij)}(\alpha_r\alpha_s + V_{rs}^{(0)}) + + 2\sum_{\nu,\nu\neq 0}k_\nu^{(ij)}\sum_{r,s=1}^{m}\Delta A_i^{(rs)}\alpha_r V_{s\nu}\cos(\nu\omega t - \vartheta_{s\nu} - \psi_\nu^{(ij)})\Big] + \xi_{j0}^{(1)} \quad (j = 1, \ldots, k) \tag{3.10}$$

Notation employed corresponds to the equations

$$V_s = \sum_{\nu,\nu\neq 0} V_{s\nu}\cos(\nu\omega t - \vartheta_{s\nu}) \tag{3.11}$$

$$\xi_{j0}^{(1)} = -\frac{1}{2}\sum_{i=1}^{k}\sum_{\nu,\nu\neq 0}k_\nu^{(ij)}\sum_{r,s=1}^{m}\Delta A_i^{(rs)}V_{rs}^{(\nu)}\cos(\nu\omega t - \vartheta_{rs}^{(\nu)} - \psi_\nu^{(ij)})$$

$$V_r V_s = V_{rs}^{(0)} + \sum_{\nu,\nu\neq 0} V_{rs}^{(\nu)}\cos(\nu\omega t - \vartheta_{rs}^{(\nu)}) \tag{3.12}$$

and $\xi_{j0}^{(1)}$ denotes the parts of ξ_{j0} independent of $\alpha_1, \ldots, \alpha_m$ and such that $\langle \xi_{j0}^{(1)} \rangle = 0$.

Substituting ξ_{j0} from (3.10) into

$$q_{r0}^{\cdot} = \sum_{s=1}^{m}\Big[A_\bullet^{(rs)} + \sum_{j=1}^{k}\Delta A_j^{(rs)}\xi_{j0}\Big]p_{s0} \tag{3.13}$$

and averaging, we obtain the following algebraic equations defining the parameters of the generating solution

$$P_r(\alpha_1, \ldots, \alpha_m) \equiv \sum_{s,u,v=1}^{m} a_{rsuv}\alpha_s\alpha_u\alpha_v + \sum_{s=1}^{m} a_{rs}\alpha_s - c_r = 0 \quad (r = 1, \ldots, m) \tag{3.14}$$

where

$$a_{rsuv} = -\frac{1}{2}\sum_{i,j=1}^{k}\Delta A_j^{(rv)}\Delta A_i^{(su)}k_0^{(ij)}, \qquad c_r = e_r - \sum_{s=1}^{m}\sum_{i=1}^{k}\Delta A_i^{(rs)}\langle \xi_{i0}^{(1)}V_s\rangle$$

$$a_{rs} = A_\bullet^{rs} + \sum_{u,v=1}^{m} a_{rsuv}V_{uv}^{(0)} - \frac{1}{2}\sum_{i,j=1}^{k}\sum_{u,v=1}^{m}\sum_{\nu,\nu\neq 0}\Delta A_j^{(ru)}\Delta A_i^{(sv)} \times \times V_{u\nu}V_{v\nu}k_\nu^{(ij)}\cos(\vartheta_{u\nu} - \vartheta_{v\nu} - \psi_\nu^{(ij)}) \tag{3.15}$$

We assume that the nonpotential forces in the coordinates of the oscillating system are the viscous friction forces. To find the condition of existence of the integral criterion, we shall construct the following derivatives:

$$\frac{\partial P_r}{\partial \alpha_s} = \sum_{u,v=1}^{m}(a_{rsuv} + a_{rusv} + a_{ruvs})\alpha_u\alpha_v + a_{rs} \tag{3.16}$$

Using the reciprocity relationships of the static action coefficients $k_0^{(ij)} = k_0^{(ji)}$ and the obvious equalities $\Delta A_j^{(rs)} = \Delta A_j^{(sr)}$ we can show that the coefficients a_{rsuv} remain unaltered if the extreme or middle indices are interchanged and also if in the first and second pairs, the indices are simultaneously rearranged: $a_{rsuv} = a_{vsur} = a_{rusv} = a_{srvu} = \ldots$ For example,

$$a_{rsuv} = -\frac{1}{2}\sum_{i,j=1}^{k} \Delta A_j^{(rv)} \Delta A_i^{(su)} k_0^{(ij)} = -\frac{1}{2}\sum_{i,j=1}^{k} \Delta A_j^{(su)} \Delta A_i^{(rv)} k_0^{(ji)} = a_{srvu} \tag{3.17}$$

Thus the sum in (3.16) is not affected by the interchange of r and s. The same property is possessed by the first two terms in the expression for a_{rs} (3.15). For the last term, using the reciprocity properties $k_\nu^{(ij)} = k_\nu^{(ji)}$ and $\psi_\nu^{(ij)} = \psi_\nu^{(ji)}$ and rearranging the indices i, j and u, v in the appropriate manner, we obtain

$$a_{rs} - a_{sr} = -\frac{1}{2}\sum_{i,j=1}^{k}\sum_{u,v=1}^{m}\sum_{\nu,\nu\neq 0} \Delta A_j^{(ru)} \Delta A_i^{(sv)} V_{u\nu} \times$$

$$\times V_{v\nu} k_\nu^{(ij)} [\cos(\vartheta_{u\nu} - \vartheta_{r\nu} - \psi_\nu^{(ij)}) - \cos(\vartheta_{v\nu} - \vartheta_{u\nu} - \psi_\nu^{(ij)})] \quad (r, s = 1, \ldots, m) \tag{3.18}$$

so that $a_{rs} = a_{sr}$ if $\vartheta_{u\nu} = \vartheta_{v\nu}$ $(u,\ v = 1,\ldots,m)$. Consequently $\partial P_r / \partial \alpha_s = \partial P_s / \partial \alpha_r$ and $P_r = \partial \Lambda / \partial \alpha_r$ in the case when the generalized forces of the first kind are component-wise phase-coupled in the quasicyclic coordinates.

The same condition ensures the following equalities

$$\left\langle \sum_{s=1}^{n-m} Q_{m+s0} \frac{\partial q_{m+s0}}{\partial \alpha_r} \right\rangle \equiv \left\langle -\left(Bu_0^{\cdot},\ \frac{\partial u_0}{\partial \alpha_r}\right) \right\rangle = b_r \qquad (r = 1, \ldots, m) \tag{3.19}$$

Here the symmetric $(n - m) \times (n - m)$ matrix B characterizes the friction in the oscillating system and b_r is independent of $\alpha_1, \ldots, \alpha_m$.

Indeed, in this case the driving forces

$$F_0 = -\frac{1}{2}\sum_{j=1}^{k}\sum_{r,s=1}^{m} \Delta A_j^{(rs)} \left[\alpha_r \alpha_s + 2\alpha_r \sum_{\nu,\nu\neq 0} V_{s\nu} \times \cos(\nu\omega t - \vartheta_\nu) + V_r V_s\right] v_j \tag{3.20}$$

generate oscillations of the form

$$u_0 = \sum_\nu u_\nu^{(1)} \cos(\nu\omega t - \vartheta_\nu) + u_{0\nu}^{(2)} \sin(\nu\omega t - \vartheta_\nu) + u_0^{(1)}, \quad \text{так как} \tag{3.21}$$

where ϑ_ν and $u^{(1)}{}_0$ are independent of $\alpha_1,\ldots,\alpha_m$ and $u^{(0)}{}_0$ is a linear form of α_r.

From this, using (3.19), we obtain

$$\left\langle \left(Bu_0^{(0)};\ \frac{\partial u_0^{(0)}}{\partial \alpha_r}\right) \right\rangle = 0 \tag{3.22}$$

With B symmetric matrix, integration by parts yields

$$W_{Q0} = \langle -(Bu^{\cdot}{}_0,\ u_0) \rangle = \langle (Bu^{\cdot}{}_0,\ u_0) \rangle = 0 \tag{3.23}$$

The linearity of ΔT_1 in ξ_j yields the following relation between the virial of the driving forces and the additional energy of the exciters:

$$W_{F_e} = (\Delta T_1)_0 \tag{3.24}$$

and the function Λ can therefore be written as (see 1.43))

$$\Lambda = \langle T_1^* \rangle_0 + {}^1/_2 \langle \Delta T_1 \rangle_0 - 2A, \quad A = \frac{1}{2}\sum_{r=1}^{m} \alpha_r(e_r - b_r) \tag{3.25}$$

or as

$$\Lambda = \langle T_1 \rangle_0 - {}^1/_2 \langle \Delta T_1 \rangle_0 - 2A \tag{3.26}$$

Two above relations together with the most general representation of Λ in terms of the averaged Routh's kinetic potential

$$\Lambda = \langle T_1 \rangle_0 - \langle L_2 \rangle_0 - 2A \tag{3.27}$$

yield three formulations of the integral criterion of stability, and two out of four following functions, namely T_1, T_1^*, ΔT_1 and L_2, are used in each formulation.

Scalar multiplying the expression for the coordinates of the oscillating system (1.36) written in the generating approximation by u_0, we obtain (cf. (1.41))

$$\langle \Delta T_1 \rangle_0 = 2 \langle L_2 \rangle_0 \tag{3.28}$$

Function Λ now becomes a sum of the ternary, quadratic and linear form of $\alpha_1, \ldots, \alpha_m$ given by

$$\Lambda = \frac{1}{4} \sum_{r,s,u,v=1}^{m} a_{rsuv}\alpha_r\alpha_s\alpha_u\alpha_v + \frac{1}{2} \sum_{r,s=1}^{m} a_{rs}\alpha_r\alpha_s - \sum_{r=1}^{m} c_r\alpha_r + \Lambda_1 \tag{3.29}$$

where Λ_1 denotes the part of Λ which is independent of $\alpha_1, \ldots, \alpha_m$ (when Λ is computed according to (3.25) - (3.27)).

In the present case each of the following two conditions is sufficient for the integral criterion to hold. First of these conditions is that nonpotential forces acting over the coordinates of the oscillating system are absent, and the other is that the nonpotential forces in the oscillating system are the viscous friction forces, while the generalized forces of the first kind defined on the quasicyclic coordinates satisfy the condition of the component-wise phase-coupling. We should, however, note that in the latter case the componentsof the vector u_0 (the coordinate of the oscillating system) will not, generally speaking, be component-wise phase-coupled and the equality of (2.19) will be conditional on the symmetry of the matrix B (when $q_{m+1}, \ldots, q_n$ are component-wise phase-coupled, then (1.17) holds for any B; ; see Sect. 1).

Coefficients a_{rsuv}, a_{rs} etc. can be obtained without use of their representations in terms of the Fourier coefficients (3.15). We can e.g. adopt the following procedure. Let us introduce the frequency-impulse matrix characteristic $K(t) = \|K^{(ij)}(t)\|$, $i, j = 1, \ldots, k$, of the oscillating system, defining it as follows [7]. Let the oscillating system be acted upon by a single $2\pi/\omega$-periodic load of the form $f(t)\, v_j$. Then the law of variation of the i-th feedback parameter with time can be given by

$$\xi_i^{(j)}(t)=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}K^{(ji)}(t-\tau)\,f(\tau)\,d\tau \tag{3.30}$$

where $K^{(jl)}$ is independent of $f(t)$. This enables us to write (3.10) as

$$\xi_{j0}=\frac{\omega}{2\pi}\sum_{i=1}^{k}\int_0^{2\pi/\omega}K^{(ji)}(t-\tau)\,F_{i0}(\tau)\,d\tau \tag{3.31}$$

and the expression for $\langle\Delta T_1\rangle_0$ as

$$\langle\Delta T_1\rangle_0=-\frac{\omega^2}{4\pi^2}\sum_{i,\,j=1}^{k}\int_0^{2\pi/\omega}\int_0^{2\pi/\omega}F_{j0}(t)\,K^{(ij)}(t-\tau)\,F_{i0}(\tau)\,d\tau\,dt \tag{3.32}$$

The previous expressions are obtained from (3.31) and (3.32) by utilizing

$$K^{(ij)}(t)=2\sum_{\nu,\nu\neq 0}k_\nu^{(ij)}\cos(\nu\omega t-\psi_\nu^{(ij)})+k_0^{(ij)} \tag{3.33}$$

Systems with "purely attractive electromagnets" in the problem on oscillations of the current-carrying conductors, correspond to the case discussed in Subsect. 2, Sect. 3. Describing these systems we can assume that, when the magnetic fluxes are given, then the field energy does not depend on the displacements within the object and is proportional to the change in the induction line length outside the object. Oscillations caused by the forces of attraction between two halves of a thin, ferromagnetic torus separated by narrow slits normal to the axis can serve as an example of such a system. The field is generated by windings on the tore connected to the given emf.

In the cases differing from those discussed in Subsect. 1 and 2, Sect. 3, equations obtained for the position coordinates in the generating approximation are nonlinear and describe the oscillations of a system acted upon by the forces depending on the displacements, time and the parameters $\alpha_1,\ldots,\alpha_m$. The energetic criteria indicated above are particularly useful here for the following reasons. Let us suppose that we can find (e.g. by numerical integration) the functions $q_{m+10},\ldots,q_{n0}$ when any $\alpha_1,\ldots,\alpha_m$ are given. The usual procedure would consist of going through the values of α_r in order to obtain approximate relationships $q_{m+s0}(t,\alpha_1,\ldots,\alpha_m)$ or, of assigning some values to α_r to find the functions q_{m+s0} and the values of $P_1,\ldots,P_m$ selecting those values of α_s for which $P_r\approx 0$. With the function Λ available, we can go through the values of α_r using well known methods of obtaining a minimum, and this shortens the computations considerably.

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Translated by L.K.

ON OPTIMIZATION OF THE TRACKING PROCESS

PMM Vol. 33, No. 1, 1969, pp. 101-111
F.L. CHERNOUS'KO
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(Received September 3, 1968)

The present paper concerns the optimization of the tracking process (with allowance for measurement errors) in a system whose motion is described by linear differential equations. It is shown that under certain assumptions the problem reduces to one of ordinary optimal control. Further analysis using the maximum principle enables us to reduce the initial problem to a system of transcendental equations. Examples illustrating optimal tracking strategy in specific cases are discussed.

Problems of optimal control in the absence of complete information, i.e. with incomplete and inexact measurements or observations, are of great interest in control engineering. Various approaches to optimal control and tracking problems with incomplete information are considered, for example, in [1-8], whose authors employ both probabilistic and minimax formulations.

1. The initial relations. Let the state of a system at any instant be defined by an n-dimensional phase coordinate vector x. The law of variation of $x(t)$ takes the form of a determinate linear system of ordinary differential equations,

$$dx / dt = A(t)x + b(t) \tag{1.1}$$

where A is an $n \times n$ matrix and b is an dimensional vector. System (1.1) can be regarded in many cases as a system in variations near the theoretical (nominal) trajectory of the initial nonlinear system.

The motion of the system is considered over the time interval $[t_0, T]$; the phase coordinates of the system are observed (measured) at the fixed instants $t_0, t_1, \ldots, t_N = T$. Here $t_k < t_{k+1}$ for $k = 0,1,\ldots,N-1$. By "observation" at each instant of time t_k we mean the approximate measurement of certain linear combinations of the components of the vector $x(t_k)$, i.e. measurement of the vector $Q_k x(t_k)$. Here Q_k is a given rectangular matrix with l_k rows and n columns. The integer $l_k \geqslant 0$ is the number of scalar parameters measured at the instant t_k, $k = 0,1,\ldots,N$. We assume that the error of each observation is a random l_k-dimensional vector quantity distributed according to a normal law with zero mathematical expectation and a known correlation matrix B_k. The term "correlation matrix" is used throughout the present paper to refer to an unnormalized correlation matrix (a second-moment matrix). The measurement error at a given instant is assumed to be independent of the errors at the other instants.

Thus, the result of observation at the instant t_k is a random l_k-dimensional vector quantity y_k with a normal distribution law. Its mathematical expectation is equal to the true value of $Q_k x(t_k)$, and its $l_k \times l_k$ correlation matrix is known and equal to B_k.